

# The large connectivity limit of bootstrap percolation

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**Abstract** – Bootstrap percolation provides an emblematic instance of phase behavior characterised by an abrupt transition with diverging critical fluctuations. This unusual hybrid situation generally occurs in particle systems in which the occupation probability of a site depends on the state of its neighbours through a certain threshold parameter. In this paper we investigate the phase behavior of the bootstrap percolation on the regular random graph in the limit in which the threshold parameter and lattice connectivity become both increasingly large while their ratio  $\alpha$  is held constant. We find that the mixed phase behavior is preserved in this limit, and that multiple transitions and higher-order bifurcation singularities occur when  $\alpha$  becomes a random variable.

**Introduction.** – Bootstrap percolation (BP) is a generalization of the ordinary random percolation problem to particle systems in which the occupation probability of a site depends on the state of its neighbours [1]. It was introduced in the late 1970s in connection with the behavior of some dilute magnets in which, under certain circumstances, an atom displays a magnetic moment only if the number of its magnetic neighbors is above a certain threshold. Since physical details of atoms and interactions are completely stripped away, BP naturally lends itself to quite distinct possible interpretations. Indeed, perhaps not surprisingly, threshold models of cooperative behavior closely related to BP have been considered early in the sociological literature [2] and have been applied to a wide range of problems (see, for a list of examples, Ref. [1]).

The solution of BP on the Bethe lattice [3] showed the presence of a peculiar phase transition in which a sudden jump of the order parameter is accompanied by diverging fluctuations as in ordinary critical phenomena. For this reason the BP transition is said to have a mixed or hybrid nature. Interest in mixed phase behavior has grown in the last years while it has been recognized that its occurrence in apparently unrelated problems, such as slow relaxation in supercooled liquids [4, 5], jamming in granular materials [6, 7], and hardness in combinatorial optimization problems [8, 9] may have a common microscopic origin [10–12], and be generally relevant to the vulnerability of complex networks under damage [13].

In this paper, we consider the BP problem on the regular random graph in the limit in which the threshold parameter and lattice connectivity become both increasingly large while holding their ratio constant. This type of mean-field limit is interesting because it offers the possibility to get extra analytical insights into the connectivity and threshold dependence of relevant quantities such as the critical point and the jump of the order parameter at the transition. We find, for example, that mixed phase behavior is generally robust, and that the critical amplitude of the order parameter decreases as a power-law of connectivity with an exponent of  $-1/4$ . This latter feature implies that the infinite connectivity limit is, somewhat surprisingly, singular. We also show that this framework is general enough to deal with complex phase behaviors which arise when the nodes of the random graph may take different values of the ratio of threshold parameter to connectivity. Depending on the relative fractions of the different type of nodes one can then observe higher-order bifurcations and multiple phase transitions [14–18], as we shall see.

**The model.** – Consider a lattice system in which every site is first randomly occupied by a particle with probability  $p$ . Then, remove from the lattice every particle which is surrounded by a number of neighbouring particles less than  $m$ . Iterating this procedure leads to two possible asymptotic results [3]: either the lattice becomes completely empty or there is a residual cluster of particles

in which every particle has at least  $m$  neighboring particles. On tree-like structure the residual cluster is infinite and the BP problem can be solved exactly, in particular one can determine the initial critical particle density  $p_c$  above which a spanning cluster appears. When  $m = 1, 2$ , random percolation (with and without dangling ends) is recovered. In this case, the phase transition is continuous and the structure of the incipient spanning cluster is fractal. We focus here on the case  $m > 2$ . We consider a Bethe lattice with fixed connectivity, specifically, a regular random graph with coordination number  $z = k + 1$ . Let us call  $B$  the probability that a site is connected to the infinite cluster through a nearest neighbour.  $B$  verifies the self-consistent equation [3]

$$B = p \sum_{i=m-1}^k \binom{k}{i} B^i (1-B)^{k-i}, \quad (1)$$

and in the following it will play the role of the order parameter [19]. It is well known [3] that when  $m > 2$  the trivial solution of Eq. (1),  $B = 0$ , becomes unstable above a critical point  $p_c$  that depends on both the connectivity and threshold. The new solution emerges discontinuously at  $p_c$  with a jump that behaves near the transition point as  $B - B_c \sim \sqrt{p - p_c}$ , and therefore critical fluctuations of the order parameter are expected from the divergence of  $dB/dp$  when  $p \rightarrow p_c^+$ . The geometric nature of critical fluctuations comes from the divergence of the mean size of *corona* clusters, i.e., clusters in which every particles is surrounded by exactly  $m$  neighbouring particles [13]. This marginality condition implies that the corona structure is extremely fragile: when a corona particle is removed the nearby particles become unstable under the BP rule. This triggers a cascade process which, by a domino-like effect, leads to the complete destruction of the corona [13].

The BP can be generalised by attaching to every node of the random graph its own threshold parameter. In this way one can explore a larger variety of situations in which, e.g., particles may have different sizes or shapes and, more generically, individuals are characterised by distinct (threshold) behaviors. Due to the competition among clusters with more or less compact structures, additional interesting features emerge in this heterogeneous case, such as multicritical like point and multiple phase transitions [14–18]. In the following, we shall be essentially concerned with phase behavior of mixed nature.

**The large connectivity limit.** – The binomial structure of Eq. (1) leads naturally to the possibility of considering the limit in which the threshold parameter and lattice connectivity are both large,  $m \gg 1$  and  $k \gg 1$ , while their ratio  $\alpha = m/k$  is a constant,  $0 < \alpha < 1$ . In this limit, according to the de Moivre-Laplace formula, each term of the sum on the right hand side of Eq. (1) tends to a Gaussian probability density function:

$$G(i) = \frac{1}{\sqrt{2\pi k B(1-B)}} \exp \left[ -\frac{(i - Bk)^2}{2kB(1-B)} \right], \quad (2)$$

with average  $\mu = k B$  and variance  $\sigma^2 = k B(1-B)$ . Passing to the continuous limit, the sum in Eq. (1) is converted to an integral and we can finally write down:

$$B = \frac{p}{2} (\operatorname{erf} Y_1 - \operatorname{erf} Y_\alpha), \quad (3)$$

where  $Y_1 = Y_{\alpha=1}$  and

$$Y_\alpha(B) = \frac{\sqrt{k}(\alpha - B)}{\sqrt{2B(1-B)}}. \quad (4)$$

We first observe that Eq. (3) has always the trivial solution  $B = 0$  because the argument of the two error functions both diverge (regardless the value of  $k$ ). Additional solutions emerging discontinuously at the critical point  $p_c(k)$  are obtained as follows. For  $k \rightarrow \infty$  we observe that the right hand side of Eq. (3) is either zero or  $p$  depending on whether  $B$  is smaller or larger than  $B_c(\infty) = \alpha$ . Therefore, we get for the critical point  $p_c(\infty) = \alpha$ , and thus  $B - B_c(\infty) = [p - p_c(\infty)]^\beta$ , with  $\beta = 1$ . As we shall see, this linear behavior is however singular, i.e., it strictly holds only for  $k = \infty$ . For any large but finite  $k$  the nature of the phase transition has generally a mixed character,  $B - B_c(\alpha, k) \sim [p - p_c(\alpha, k)]^\beta$  with  $\beta = 1/2$ , over a sufficiently small region near the critical point, provided that  $\alpha$  is neither zero nor one.

*The fold bifurcation.* To determine the location of the discontinuous critical point for finite yet large  $k$  we proceed as follows. We first rewrite the self-consistent equation for  $B$ , eq. (3) as

$$\frac{1}{p} = \mathcal{F}(B) \quad (5)$$

where

$$\mathcal{F}(B) = \frac{\operatorname{erf} Y_1 - \operatorname{erf} Y_\alpha}{2B} \simeq \frac{\operatorname{erfc} Y_\alpha}{2B}. \quad (6)$$

Then, since we are seeking a discontinuous solution with critical fluctuations (i.e., with  $\beta < 1$ ), we set  $\mathcal{F}'(B) \sim dp/dB$  to zero to get:

$$\operatorname{erfc} Y_\alpha \simeq -\frac{2B}{\sqrt{\pi}} \frac{dY_\alpha}{dB} \exp(-Y_\alpha^2). \quad (7)$$

To find a closed expression for  $B_c(\alpha, k)$  we approximate

$$Y_\alpha \simeq \frac{\sqrt{k}(\alpha - B)}{\sqrt{2\alpha(1-\alpha)}}, \quad \frac{dY_\alpha}{dB} \simeq \frac{-\sqrt{k}}{\sqrt{2\alpha(1-\alpha)}}, \quad (8)$$

and use the asymptotic expansion of the complementary error function for large negative values of its argument:

$$\operatorname{erfc}(x) \simeq 2 + \frac{e^{-x^2}}{x\sqrt{\pi}}. \quad (9)$$

Plugging this expression in Eq. (3) we obtain to the leading order in  $k$ :

$$B_c(\alpha, k) \simeq \alpha - \sqrt{\frac{\alpha(1-\alpha)}{k} \ln \frac{k\alpha}{2\pi(1-\alpha)}}. \quad (10)$$

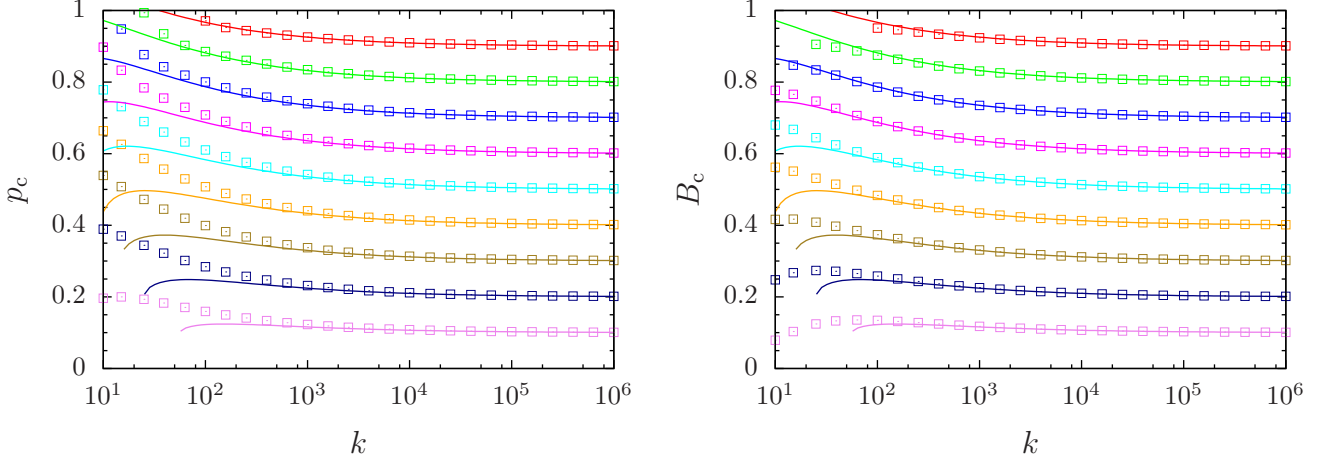


Fig. 1: Scaling of the critical point,  $p_c$ , and the order parameter,  $B_c$ , with the random graph connectivity  $k$  for  $\alpha$  going from 0.1 to 0.9 with a step of 0.1 (from the bottom to the top). Square symbols denote the exact solution of Eq. (3) while full lines represent the approximated analytical behavior given by Eqs. (10) and (11).

Correspondingly, in this approximation the phase transition point is located at

$$p_c(\alpha, k) \simeq B_c(\alpha, k). \quad (11)$$

In the Fig. 1 Eqs. (10) and (11), are compared with the exact solution as a function of  $k$  and for several values of  $\alpha$ . The asymptotic approximation appears to work pretty well when  $k$  is generally larger than a few hundreds. In this region  $B_c(\alpha, k)/\alpha$  depends solely on the scaled variable  $k\alpha/(1-\alpha) \simeq \mu^2/\sigma^2$ , as can be easily argued from Eq. (10).

To find the behavior of the  $B(p)$  near the critical point we take the derivative of the Eq. (5):

$$\frac{dp}{dB} \sim \sqrt{k}(B - B_c). \quad (12)$$

Close enough to the critical point,

$$p - p_c \sim \sqrt{k}(B - B_c)^2, \quad (13)$$

which finally gives, to the leading order in  $k$ :

$$B - B_c \sim k^{-1/4} \sqrt{p - p_c}. \quad (14)$$

We obtain therefore the well known square-root dependence of the order parameter jump from the distance to the transition point which characterises the singular behavior near a fold bifurcation. Notice that in our mean-field limit the critical amplitude of the jump decrease as a power-law of the connectivity with an exponent  $1/4$ . These behaviors perfectly agree with the exact solution of Eq. (3) as shown in the Fig. 2. When we are sufficiently deep in the critical region the order parameter depends linearly on the distance from the critical point, see Fig. 2. Comparing the linear and the square-root dependence one can thus find the crossover point between these two behaviors and get that the range of the square-root dependence decreases with the connectivity as  $k^{-1/2}$ .

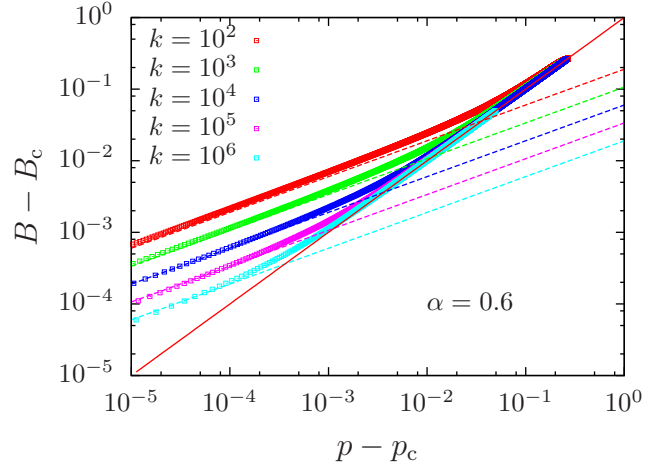


Fig. 2: Order parameter jump,  $B - B_c$ , vs distance to the critical point,  $p - p_c$ , for several values of  $k$ , with  $\alpha = 0.6$ . The dashed straight lines are proportional to  $k^{-1/4} \sqrt{p - p_c}$ . The full line with slope one represents the linear behavior regime.

*Multiple transitions and higher bifurcations.* The fold bifurcation discussed above, which is denoted with  $A_2$  in the Arnold's terminology, represents only the lowest level of the hierarchy of bifurcation singularities [20]. It has been recently observed that the formal structure of the self-consistent equation obeyed by  $B$  in BP is essentially similar to that satisfied by the nonergodicity parameter in the mode-coupling theory of the liquid-glass transition [18]. Consequently, one can devise suitable generalizations of the BP problem which provide a microscopic realization of the hierarchy of bifurcations in close analogy with that appearing in some glassy systems [18].

We remind that an  $A_\ell$  bifurcation of order  $\ell$  corresponds

to the maximum root of Eq. (5) which simultaneously satisfies the equations:

$$\frac{d^n \mathcal{F}}{dB^n} = 0, \quad n = 1, \dots, \ell - 1; \quad \frac{d^\ell \mathcal{F}}{dB^\ell} \neq 0. \quad (15)$$

Bifurcation singularities are interesting because they have distinct critical properties. For example, the Taylor expansion of  $\mathcal{F}$  near the bifurcation surface and Eqs. (15), imply that the order parameter scales like  $(p - p_c)^{1/\ell}$  near an  $A_\ell$  singularity. This means that order parameter fluctuations are stronger when the order of the bifurcation singularity increases.

A generalization of BP in which higher-order bifurcations and multiple transitions are possible can be obtained by letting the  $\alpha$  parameter depend on the lattice sites. Notice that in the presence of multiple roots of Eq. (5) one has always to select the maximum one because  $B$  is a monotonically increasing function of  $p$ . Here we consider the case in which  $\alpha$  is randomly distributed as

$$\mathcal{P}(\alpha) = \sum_{i=1}^n q_i \delta(\alpha - \alpha_i), \quad \sum_{i=1}^n q_i = 1, \quad (16)$$

with  $q_i$  being the fraction of sites having threshold  $\alpha_i$ . In this situation the equation for  $B$  becomes

$$B = \frac{p}{2} \left( \operatorname{erf} Y_1 - \sum_{i=1}^n q_i \operatorname{erf} Y_{\alpha_i} \right). \quad (17)$$

Hereafter, we focus exclusively on binary mixtures of threshold parameters,  $n = 2$ , with  $0 < \alpha_i < 1$ . We note, however, that it is a simple matter to extend our calculations to  $n$ -ary mixtures and to situations in which one of the threshold parameters approaches one. In the latter limit one recovers the ordinary random percolation which may be interesting for those systems in which the interplay of continuous and discontinuous percolation transitions is relevant.

*Infinite connectivity.* We now first look at the phase structure of a binary mixture in the limit  $k \rightarrow \infty$  which will provide a guideline for the case of large  $k$ . We use  $p$  and  $q_2$  as control variables. Without loss of generality we assume for sake of simplicity that  $\alpha_1 \geq \alpha_2$ , and distinguish three regions for the order parameter.

- i) When  $\alpha_1 \geq \alpha_2 \geq B$  one has  $\operatorname{erf} Y_{\alpha_1} = \operatorname{erf} Y_{\alpha_2} = 1$  and thus  $B = 0$ , which corresponds to the empty phase where every particle of the initial configuration can be removed by using the BP rule.
- ii) For  $B \geq \alpha_1 \geq \alpha_2$  we have  $\operatorname{erf} Y_{\alpha_1} = \operatorname{erf} Y_{\alpha_2} = -1$  which implies  $B = p > 0$  meaning that there is a finite fraction of particles which cannot be removed from the system. The phase boundary is obtained when  $B = \alpha_1$ , and is therefore a horizontal line in the plane  $(p, q_2)$  located at

$$p_{c_1} = \alpha_1. \quad (18)$$

This transition line is properly defined in the range  $q_2 \in [0, 1)$ , because in the limit  $q_2 \rightarrow 1$  we have to recover correctly the pure BP problem which has a transition located at  $p_{c_2} = \alpha_2$  (as the fraction  $q_1$  of type-1 sites is zero). The region  $p \geq p_{c_1}$  will be denoted as BP<sub>1</sub> phase and is depicted in the Fig. 3a.

- iii) Finally, when  $\alpha_1 \geq B \geq \alpha_2$  we get  $\operatorname{erf} Y_{\alpha_1} = 1$  and  $\operatorname{erf} Y_{\alpha_2} = -1$  and therefore  $B = pq_2$ . The phase boundary with the empty phase corresponds to  $B = \alpha_2$ , and is located at:

$$p_{c_2} = \frac{\alpha_2}{q_2}. \quad (19)$$

The region immediately above the hyperbolic branch, Eq. (19), will be denoted as BP<sub>2</sub> phase, see Fig. 3a. This loose phase is intermediate between the empty phase and the more compact BP<sub>1</sub> one: it appears as soon as the fraction  $q_2$  of type-2 sites, becomes sufficiently large to support a percolating structure. Notice that the stability limit of Eq. (19) lies above the crossover point with the first transition line, Eq (18), which is given by  $p_{c_1} = p_{c_2}$ , that is:

$$q_{cr} = \frac{\alpha_2}{\alpha_1}, \quad (20)$$

otherwise  $B$  would be a decreasing function of  $p$ , which is uncorrect. Therefore, this second transition line is limited to the range  $q_2 \geq q_{cr}$ . Notice that, as long as  $q_1$  is nonzero, the passage from BP<sub>2</sub> to BP<sub>1</sub> always involves the crossing of the boundary between the two percolating phases, i.e., a discontinuous transition. As we shall see, this is a peculiar feature of the infinite connectivity limit.

*Large connectivity.* The phase behavior for large  $k$  can be obtained by following closely the steps outlined above for the infinite connectivity case. We first approximate

$$\frac{1}{p} = \mathcal{F}(B) \simeq \frac{1}{2B} \sum_{i=1}^2 q_i \operatorname{erfc} Y_{\alpha_i}, \quad (21)$$

and setting  $\mathcal{F}'(B) = 0$  we get:

$$\sum_{i=1}^2 q_i \left[ \operatorname{erfc} Y_{\alpha_i} + \frac{2B}{\sqrt{\pi}} Y'_{\alpha_i} e^{-Y_{\alpha_i}^2} \right] = 0. \quad (22)$$

We then distinguish two regions in which either  $B \geq \alpha_1 \geq \alpha_2$  or  $\alpha_1 \geq B \geq \alpha_2$  and observe that when  $\alpha_1$  and  $\alpha_2$  are not too close to each other, the Eq. (22) is essentially equivalent to two independent equations corresponding to the phase boundaries in which either  $B \approx \alpha_1$  or  $B \approx \alpha_2$ .

In the former case,  $B \approx \alpha_1$ , we have  $\operatorname{erfc} Y_{\alpha_1}(B) \simeq \operatorname{erfc} Y_{\alpha_2}(B) \simeq 2$  and neglecting the term containing  $e^{-Y_{\alpha_2}^2}$  in Eq. (22), leads to:

$$1 \simeq q_1 \frac{B}{\sqrt{\pi}} Y'_{\alpha_1} e^{-Y_{\alpha_1}^2}. \quad (23)$$

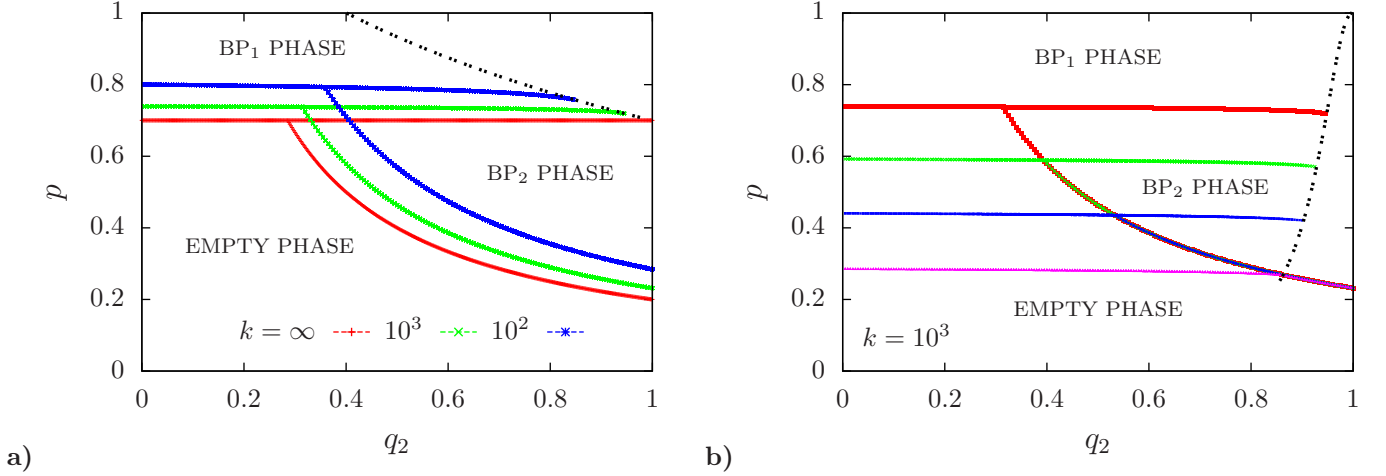


Fig. 3: Two sections of the phase diagram of generalised BP in the control variables  $p$  and  $q_2$ . The double-dotted lines in either figures represent cuts in the surface of  $A_3$  singularities. **a)**  $\alpha_1 = 0.7$  and  $\alpha_2 = 0.2$  for three values of  $k$ . **b)**  $k = 10^3$ ,  $\alpha_2 = 0.2$  and  $\alpha_1 = 0.7, 0.55, 0.4, 0.25$  from top to bottom; the  $A_4$  bifurcation corresponds to the terminal point of the double-dotted line.

Thus, the first transition line delimiting the  $BP_1$  phase is:

$$p_{c_1} = \frac{2B_{c_1}}{q_1 \operatorname{erfc} Y_{\alpha_1}(B_{c_1}) + q_2 \operatorname{erfc} Y_{\alpha_2}(B_{c_1})}, \quad (24)$$

with

$$B_{c_1} \simeq \alpha_1 - \sqrt{\frac{\alpha_1(1-\alpha_1)}{k} \ln \frac{q_1^2 k \alpha_1}{2\pi(1-\alpha_1)}}. \quad (25)$$

In the latter case, the phase boundary corresponds to  $B \approx \alpha_2$  and we can consider  $\operatorname{erfc} Y_{\alpha_2} \ll \operatorname{erfc} Y_{\alpha_1} \simeq 2$  in Eq. (22). Neglecting the term containing  $e^{-Y_{\alpha_1}^2}$  we get

$$2 = \frac{2B}{\sqrt{\pi}} Y_{\alpha_2}' e^{-Y_{\alpha_2}^2}. \quad (26)$$

This leads to the second transition line:

$$p_{c_2} = \frac{2B_{c_2}}{q_2 \operatorname{erfc} Y_{\alpha_2}(B_{c_2})}. \quad (27)$$

where  $B_{c_2} = B_c(\alpha_2, k)$ , with  $B_c$  being the function defined by Eq. (10).

Therefore, also for large connectivity we obtain two transition lines delimiting two percolating phases, and thus the possibility of multiple discontinuous phase transitions, see Fig. 3a for two examples with  $k = 10^2, 10^3$  (for fixed  $\alpha_1 = 0.7$  and  $\alpha_2 = 0.2$ ). Note, that also in this two-specie case the relevant scaling variables on which the two transition lines do depend are  $k\alpha_i/(1-\alpha_i)$  with  $i = 1, 2$ .

It is now interesting to observe that, at variance with the  $k = \infty$  case, the transition line  $p_{c_1}$  has a terminal endpoint corresponding to the vanishing of the logarithm term in Eq. (25), that is:

$$q_2^{A_3} = 1 - \sqrt{\frac{2\pi(1-\alpha_1)}{k\alpha_1}} \quad (28)$$

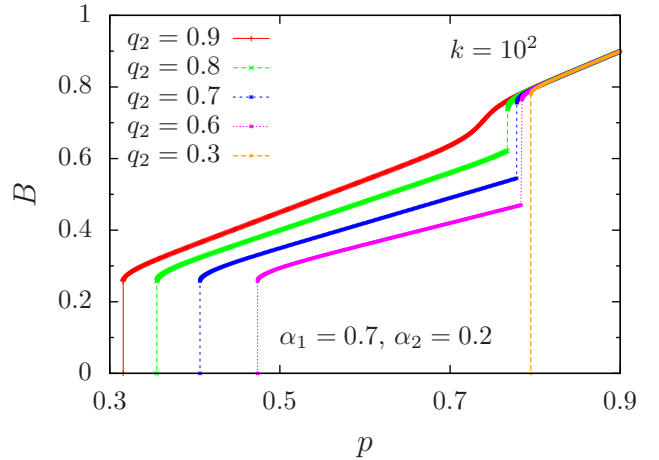


Fig. 4: Order parameter  $B$  vs the occupation probability  $p$ . For  $q_2 = 0.1$  the system passes very near to the  $A_3$  singularity, while for larger values of  $q_2$  it goes through the three phases and so the curves show a double jump.

This is essentially equivalent, to the leading order in  $k$  we consider, to the point where  $\mathcal{F}'' = 0$ , i.e., to an  $A_3$  singularity or cusp bifurcation, see Fig. 3a. The strength of the discontinuous transition between the two percolating phases weakens as the  $A_3$  critical endpoint is approached upon increasing  $q_2$  until the distinction between the  $BP_1$  and  $BP_2$  phases disappears. More generally, the presence of the critical endpoint implies that there is always a path in the control variables space which allows a smooth passage between the two percolating phases, i.e., without observing an extra jump in the order parameter. In Fig. 3a)



we also show the bifurcation line of  $A_3$  singularities:

$$p^{A_3}(q_2) = \frac{2\alpha_1}{1 - q_2 + q_2 \operatorname{erfc} \frac{\sqrt{\pi}(\alpha_2 - \alpha_1)}{(1 - q_2)\alpha_1}}. \quad (29)$$

This is obtained by expressing  $k$  in terms of  $q_2$  through Eq. (28) and plugging the obtained result in Eq. (24).

Fig. 3b) instead shows a section of the phase diagram for several values of  $\alpha_1$  at fixed connectivity  $k = 10^3$  and  $\alpha_2 = 0.2$ . The topology of the phase diagram remains essentially unchanged. In particular the  $A_3$  bifurcation line is now obtained by using  $\alpha_1(q_2)$  from Eq. (28) in Eq. (24), and there is no need to report the explicit expression. However, it is interesting to note that when  $\alpha_1$  gets closer to  $\alpha_2$  the transition line between the percolating phases shrinks until a special value of  $\alpha_1$  (generally different from  $\alpha_2$ ) is reached at which only one percolating phase survives. This occurs when the  $A_3$  critical endpoint coalesces with the crossover point between the  $p_{c_1}$  and  $p_{c_2}$  transition lines, or, equivalently, when  $p^{A_3} = p_{c_2}$ . This corresponds to the formation of an  $A_4$  singularity, also known as swallowtail bifurcation. In Fig. 3b) the terminal point of the  $A_3$  cusp line represents the  $A_4$  singularity to the leading order approximation considered here, which is almost indistinguishable from the exact result, as we generally find in the large connectivity limit. An illustration of the complex phase behavior discussed above is shown in Fig. 4 where we see the typical evolution of the order parameter near the  $A_3$  singularity and the double jump it undergoes when the  $BP_1$  and  $BP_2$  phases are crossed upon changing  $p$ .

**Conclusions.** — To summarize, the large connectivity limit of bootstrap percolation provides a suitable framework in which one can investigate analytically the dependence of critical properties on the threshold parameter and lattice connectivity. We generally found that all the interesting features of the phase structure found in low connectivity random graphs, such as hybrid phase behavior, multiple transitions and higher-order singularities, are robust in this limit, although, perhaps surprisingly, the infinite connectivity limit turns out to be singular.

For general  $n$ -ary mixtures one can reasonably speculate that the phase structure will entail  $n$  percolating phases, provided that the values of the random variable  $\alpha$  (that is, the ratio of threshold parameter to connectivity) are sparse enough. As the initial density of particles  $p$  increases, one would then observe, in a suitable domain of the control variables, a sequence of multiple transitions starting from a loose percolating phase (the one with the lowest value of  $\alpha$ ), towards more and more compact percolating phases (those with higher and higher values of  $\alpha$ ); in particular, when the lowest value of  $\alpha$  is zero the incipient percolation cluster that first appears has a fractal structure and the associated phase transition is an ordinary random percolation transition.

We expect that the present framework is especially relevant for the study of massively connected networks and

that it can be rather effective in the study of threshold dynamics closely related to BP [21]. We also remark that the large connectivity limit could be a valuable tool for addressing the dynamics of facilitated spin models of glasses which, in spite of their apparent simplicity, still lacks an analytic solution. Work in this direction is in progress.

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$$P = p \sum_{i=0}^{z-m} \binom{z}{i} B^{z-i} (1-B)^i,$$

its critical properties near the transition are the same as those of  $B$ .

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